An Optimal Maintenance Policy for a Multi-State Deteriorating System

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Introduction

- Multi-state repairable system with k failure states and 1 working state.
- System deteriorates over time and will be replaced after N failures.
- A PM is performed when system's reliability drops a critical threshold R.
- A CM is performed after each failure.
- PM and CM are imperfect according to geometric processes.
- CM durations are taken into account.

Aim of the paper : derive an optimal sequential failure limit maintenance policy (R^*, N^*) such that the long-run expected cost per unit time is minimized.

Multi-state systems

1 working state.

k failure states, classified by features such as

- failure severity (the cost related to each failure state is different),
- failure cause (the treatment related to each failure state is different),
- ...

After a failure, the system will be in the sth-type failure state with probability p_s, $\forall s$, $p_s \geq 0$ and $\sum_{s=1}^{k} p_s = 1$.

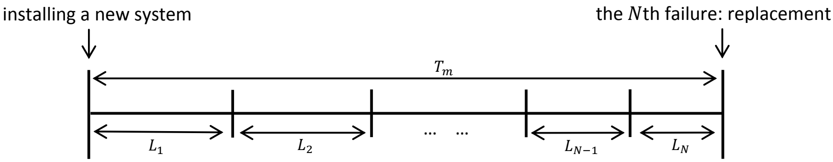
The probability of occurrence of each failure state does not change with time.

Replacement policy

At the beginning, a new system is installed.

It will be replaced upon its Nth failure by a new physically and statistically identical one.

Replacement duration is negligible.



L_n: lengths of successive repair cycles.

T_m: time between the $(m-1)$th and the mth replacement. $\{T_m\}_{m\geq 1}$ is a renewal process.

Corrective and Preventive Maintenance

Failure-limit PM policy: a PM is performed when system's reliability drops a critical threshold R.

PM durations are negligible.

A CM is performed after each failure.

CM durations cannot be neglected.

PM and CM are imperfect (not AGAN):

- In practice, the successive operating durations after repairs often show a decreasing trend.
- In practice, the successive repair durations often show an increasing trend.

The Geometric process

$\{Z_n\}_{n \geq 1}$ sequence of independent positive random variables.

Geometric process with parameter $a > 0$, **GP(a)** (Lam, 1988):

$$Z_n \text{ has the same distribution as } Z_{n-1}/a$$

or

$$Z_n \text{ has the same distribution as } Z_1/a^{n-1}$$

or

$$\forall n \geq 1,\ P(Z_n \leq z) = P(Z_{n-1} \leq az) = P(Z_1 \leq a^{n-1} z)$$

$a > 1 \Rightarrow$ stochastically decreasing GP: $\forall n,\ Z_n \leq_{st} Z_{n-1}$.

$0 < a < 1 \Rightarrow$ stochastically increasing GP: $\forall n,\ Z_n \geq_{st} Z_{n-1}$.

$a = 1 \Rightarrow$ renewal process.

Equivalent formulation: Quasi-Renewal process (Wang-Pham 1996).

PM and CM efficiency

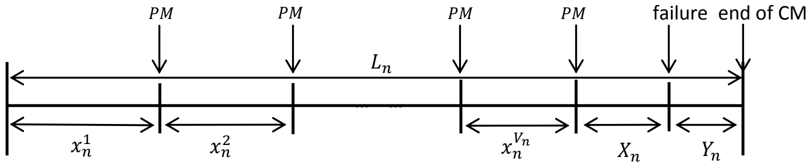
PM efficiency

- $GP(a)$ on the working time distribution, $a \geq 1$.
- $GP(b)$ on the repair duration distribution, $0 < b \leq 1$.

CM efficiency in failure state s, $s = 1, \ldots, k$

- $GP(a_s)$ on the working time distribution, $a_s \geq 1$.
- $GP(b_s)$ on the repair duration distribution, $0 < b_s \leq 1$.

nth repair cycle



- V_n = number of PM actions.
- x_n^j = inter PM working times, $j = 1, \ldots, V_n$.
- X_n = remaining useful life, working time between V_nth PM and failure.
- Y_n = duration of nth CM.

$\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ are independent.

The length of the nth repair cycle is $L_n = \sum_{j=1}^{V_n} x_n^j + X_n + Y_n$.

Reliability computations: first inter PM times and RUL

The initial lifetime Z_1 has cdf $F(t) = P(Z_1 \leq t)$.

If it happens before failure ($V_1 \geq 1$), the first PM is done at time x_1^1 such that $P(Z_1 > x_1^1) = 1 - F(x_1^1) = R$.

$$\Rightarrow x_1^1 = F^{-1}(1 - R).$$

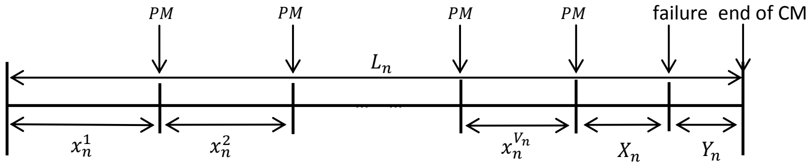
Effect of the first PM on the working time distribution according to a $GP(a)$: the remaining lifetime Z_2 has cdf $F(at)$.

$$P(Z_2 > x_1^2) = 1 - F(ax_1^2) = R \Rightarrow x_1^2 = \frac{F^{-1}(1 - R)}{a}.$$

And so on: $x_1^j = \dfrac{F^{-1}(1 - R)}{a^{j-1}}$, $j = 1 \dots V_1$.

Given $V_1 = v_1$, the remaining useful life X_1 has the same distribution as a random variable with cdf $F(a^{v_1} t)$ given that it is less than $\tau_1 = F^{-1}(1 - R)/a^{v_1}$.

Reliability computations: first failure and CM



The initial CM duration has cdf $G(t)$.

Given $V_1 = v_1$, the effect of each of the v_1 PM on the CM duration is $GP(b)$.

After the first failure, the system skips to state s_1 with probability p_{s_1}.

Given $S_1 = s_1$, the effect of failure on the CM duration is $GP(b_{s_1})$.

$\Rightarrow$ Given $V_1 = v_1$ and $S_1 = s_1$, the cdf of Y_1 is $G(b^{v_1} b_{s_1} t)$.

Reliability computations: second repair cycle

Given $V_1 = v_1$, the effect of each of the v_1 PM on the working time is $GP(a)$.

Given $S_1 = s_1$, the effect of failure on the working time is $GP(a_{s_1})$.

$\Rightarrow$ Given $V_1 = v_1$ and $S_1 = s_1$, the initial lifetime in the second repair cycle has cdf $F(a^{v_1} a_{s_1} t)$.

Given $V_2 \geq 1$, $x_2^1 = \dfrac{F^{-1}(1-R)}{a^{v_1} a_{s_1}}$.

Given $V_2 \geq 2$, $x_2^2 = \dfrac{F^{-1}(1-R)}{a^{v_1+1} a_{s_1}}$.

And so on ...

Reliability computations: nth repair cycle

Given $V_1 = v_1, S_1 = s_1, \ldots, V_{n-1} = v_{n-1}, S_{n-1} = s_{n-1}, V_n = v_n$,

- Inter PM times: $x_n^j = \dfrac{F^{-1}(1-R)}{a^{\sum_{i=1}^{n-1} v_i + j - 1} \prod_{i=1}^{n-1} a_{s_i}}$, $j = 1, \ldots, V_n$.
- Remaining useful life: the cdf of X_n is $\dfrac{F\left(a^{\sum_{i=1}^{n-1} v_i} \prod_{i=1}^{n-1} a_{s_i} t\right)}{1-R}$ for $t < \tau_n = \dfrac{F^{-1}(1-R)}{a^{\sum_{i=1}^{n-1} v_i} \prod_{i=1}^{n-1} a_{s_i}}$.
- CM duration: given $S_n = s_n$, the cdf of Y_n is $G\left(b^{\sum_{i=1}^{n} v_i} \prod_{i=1}^{n} b_{s_i}\right)$.

Reliability computations: nth repair cycle

Number of PM: $P(V_n = v_n) = R^{v_n}(1 - R)$, $\forall n$.

Failure states: $P(S_i = s_i) = p_{s_i}$, $\forall i$.

By independence,

$$P(V_1 = v_1, S_1 = s_1, \ldots, V_n = v_n) = R^{\sum_{i=1}^{n} v_i}(1 - R)^n \prod_{i=1}^{n-1} p_{s_i}$$

$\Rightarrow$ the unconditional distributions of x_i^j, X_n, Y_n can be computed.

Then it can be proved that

- $\{X_n\}_{n \geq 1}$ is a stochastically decreasing process
- $\{Y_n\}_{n \geq 1}$ is a stochastically increasing process

Cost rate function

Maintenance costs:

- C_p = PM cost.
- c_s = cost of CM in failure state s.
- c_f = cost rate per unit of time due to down time.
- C = replacement cost.

The expected cost-rate for critical threshold R and failure number N is:

$$C(R, N) = \frac{\text{expected cost incurred in a renewal cycle}}{\text{expected length of the renewal cycle}}$$

Explicit expression of $C(R, N)$

$$C(R,N) = \frac{C + NC_pR/(1-R) + c_f\psi_3 + N\sum_{s=1}^{k} c_s p_s}{\psi_1+\psi_2+\psi_3}$$

where

$$\psi_1 = \frac{aRF^{-1}(1-R)\left(1-[A(1-R)a/(a-R)]^N\right)}{(a-R)\left[1-A(1-R)a/(a-R)\right]}$$

$$\psi_2 = \frac{a\lambda(R)\left(1-[A(1-R)a/(a-R)]^N\right)}{(a-R)\left[1-A(1-R)a/(a-R)\right]}$$

$$\psi_3 = \frac{B(1-R)\mu b\left(1-[B(1-R)b/(b-R)]^N\right)}{(b-R)\left[1-B(1-R)b/(b-R)\right]}$$

with $\lambda(R) = \int_0^{F^{-1}(1-R)} tdF(t)$ $\quad \mu = \int_0^{+\infty} tdG(t)$

$A = \sum_{s=1}^{k} \frac{p_s}{a_s}$ $\quad B = \sum_{s=1}^{k} \frac{p_s}{b_s}$

Optimal maintenance policy

$$(R^*, N^*) = \underset{(R,N)}{\operatorname{argmin}}\, C(R, N)$$

Optimization procedure in two steps :

- Fix N and minimize $C(R, N)$ in R : R^*_N.
- Minimize $C(R^*_N, N)$ in N.

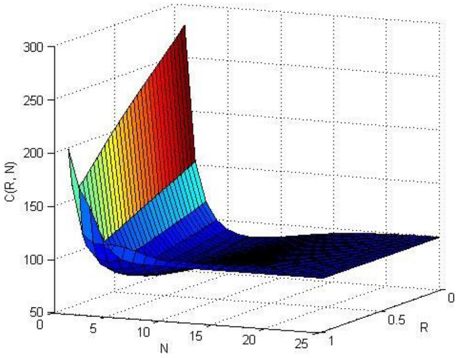
Remarks

- The initial distribution of the CM duration G is involved only through its mean μ.
- Parameters a, a_s, b, b_s should be close to 1, otherwise the estimated system reliability decreases very rapidly.

A numerical example

- The lifetime of a new system has a Weibull distribution $\mathcal{W}(\eta, \beta)$. $\eta = 2000, \beta = 1.5$.
- $k = 2$ failure states, $p_1 = 0.45, p_2 = 0.55$.
- PM effect: $a = 1.03, a_1 = 1.1, a_2 = 1.2$.
- CM effect: $b = 0.98, b_1 = 0.9, b_2 = 0.8$.
- Mean of initial CM duration: $\mu = 240$.
- Maintenance costs:
  $C_p = 5000, c_f = 100, c_1 p_1 + c_2 p_2 = 10000, C = 500000$.

Cost rate



Optimal policy : $R^* = 0.6488, N^* = 6.$

Sensitivity analysis

- When the PM cost increases, the optimal critical threshold R^* decreases.
- When the CM cost increases, the optimal critical threshold R^* increases and the optimal number of CM N^* decreases.
- When the replacement cost increases, the optimal number of CM N^* increases.

Conclusion

We have developed a bivariate maintenance policy for a multi-state deteriorating system.

- Many systems can be described under this framework.
- The maintenance strategy is very flexible, including many maintenance strategies as special cases:
  - $a = b = 1$: perfect PM.
  - $R = 0$: no PM.
  - $k = 1$: one failure state.

Prospects

- Analyze systems with multiple components and more than one working state.
- The probability occurrence of the failure states could depend on the age of the system.
- Use other imperfect maintenance models : ARA, BP,...
- Statistical issues: estimation of model parameters.
  - expert judgments: μ, a, b,
  - maximum likelihood: parameters of F.

Thank you for your attention